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Some Domain Decomposition and Iterative Refinement Algorithms for Elliptic Finite Element Problems

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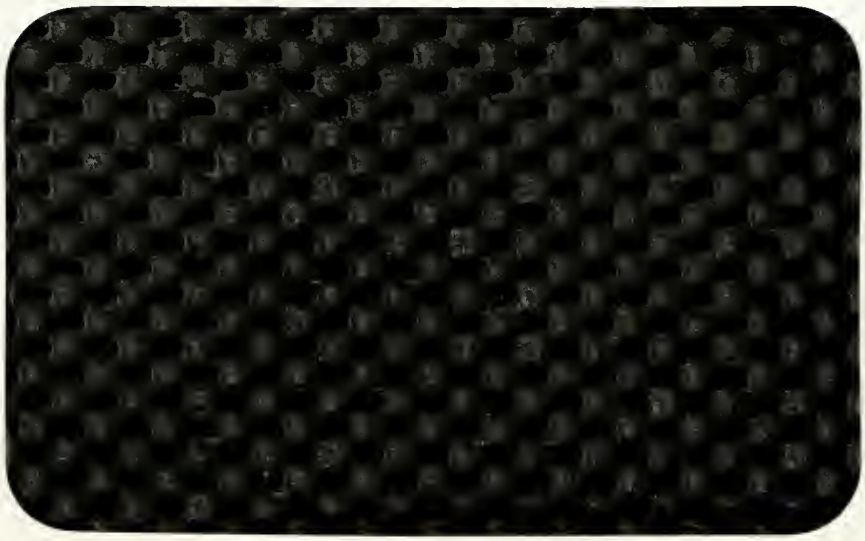

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Some Domain Decomposition and Iterative Refinement Algorithms for Elliptic Finite Element Problems

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Abstract

In this contribution, we report on some results recently obtained in joint work with Maksymilian Dryja. We first study an additive variant of Schwarz' alternating algorithm and establish that a fast method of this kind can be devised which is optimal in the sense that the number of conjugate gradient iterations required, to reach a certain tolerance, is independent of the mesh size as well as the number of subregions. An interesting feature of this algorithm is that all the subproblems can be solved at the same time. The method is therefore quite promising for parallel computers. Using a similar mathematical framework, we then consider the solution of elliptic finite element problems on composite meshes. Such problems can be built up systematically by introducing a basic finite element approximation on the entire region and then repeatedly selecting subregions, and subregions of subregions, where the finite element model is further refined. We consider conjugate gradient algorithms where, in each iteration, problems on the subregions representing finite element models prior to further refinement are solved. This makes it possible to use solvers for problems with uniform or relatively uniform mesh sizes, while the composite mesh can be strongly graded. We remark that this work is technically quite closely related to our previous work on iterative substructuring methods, which are domain decomposition algorithms using non-overlapping subregions.

1 Introduction

In this paper, we report on results obtained in joint work with Maksymilian Dryja of the University of Warsaw, Poland. Our work began with the study of a recent paper by P.-L. Lions in which a variational framework is introduced for the classical, multiplicative (sequential) Schwarz' method; see Schwarz [19] and Lions [13]. As pointed out by Lions, the variational framework is far from new; cf. [20], [2], [1] and other references given in Lions [13].

In the second section of this paper, we introduce a framework similar to Lions' and also an additive (parallel) version of the Schwarz algorithm. While Lions considered the continuous problems, we work consistently with conforming finite element approximations; cf. [6].

In view of the interest in parallel computers with many processors, we are principally interested in the case of many subregions. Our main result is that we can design additive methods of this kind which have rates of convergence which are independent of the number of subregions and the number of unknowns. No proofs are given in this paper; cf. Dryja and Widlund [8] for details. We note that the technical tools used there are similar to those of our previous work on so-called iterative substructuring methods, which are domain decomposition methods with nonoverlapping subregions; cf. e.g. [7] and [21] and the articles cited in those papers.

In the final section, we discuss optimal iterative refinement methods. This work begun with the discovery that the so called FAC and AFAC methods, which have been studied by McCormick and his coworkers [15], [14], [16], [17] and [12], and Bramble, Ewing et al. [5], [9], are structured quite similarly to the classical Schwarz procedure and its additive variant, respectively. In our view, the central theoretical issue of the iterative refinement algorithms is the design and study of algorithms for which the rate of convergence is independent of the number of subproblems, i.e. the number of refinement levels, as well as the mesh size. After introducing certain finite element models on composite meshes, which can systematically be built up inside a framework of conforming finite elements, we describe several algorithms, for which we have established optimality in this sense; cf. [23].

Let us also note that we have discovered that the very interesting so called hierarchical basis method, introduced recently by Yserentant [24], can be derived and understood inside our general framework. We plan to return to this topic elsewhere.

Since the theme of this seminar is boundary integral equations and boundary element methods, a remark on a connection between this main topic and domain decomposition algorithms might be in order. In our early work on domain decomposition algorithms for regions divided into two non-overlapping subdomains, we established that certain discrete boundary operators play a crucial role in the description and analysis of the so called Neumann-Dirichlet algorithm; cf. Bjørstad and Widlund [4]. In the basic form of this algorithm, the problems on the different subregions are solved one after the other, using natural and essential boundary conditions in alternate steps. The resulting iteration matrix can be represented in terms of so called Schur complements of the stiffness matrices of the subregions. The Schur complements are obtained by eliminating all the interior variables in the subregions. We can equally well regard them as the restrictions of the quadratic forms, which represent the energy of the subregions, to spaces of discrete harmonic functions. These boundary operators map the trace of the finite element functions onto a natural approximation of the normal derivative at the boundary of the corresponding discrete harmonic function. By using finite element techniques, we have established a bound, which is a proper analog of a standard estimate of elliptic theory. This bound is crucial in the proof that the rate of convergence of the iterative scheme is independent of the mesh size.

These operators can be regarded as discretizations of what most appropriately have been called the canonical integral operators; cf. Feng [10]. The same bound is available for these boundary integral operators and their discretizations as for the finite element operator. We can therefore apply the Neumann- Dirichlet algorithm to coupled finite element - boundary integral equation models. In particular, if the same boundary space is used for the two subproblems, we can establish that the rate of convergence of the resulting method is independent of the number of degrees of freedom.

2 An Additive Variant of the Schwarz Alternating Method

We consider linear, self adjoint, elliptic problems discretized by finite element methods on bounded Lipschitz regions. We assume that Ω is a region in R^n, that the differential operator is the Laplacian and that we use continuous, piecewise linear finite elements. The theory can equally well be developed for any linear elliptic problem which can be formulated as a minimization problem, and we can also use arbitrary conforming finite elements. The continuous and discrete problems are of the form

$$a(u,v) = f(v),\ \forall\ v \in\ V\ ,$$

and

$$a(u_h,v_h) = f(v_h),\ \forall\ v_h \in\ V^h\ , \tag{1}$$

respectively. The bilinear form is defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx\ .$$

It defines a semi-norm $|u|_{H^1} = (a(u,u))^{1/2}$ in the Sobolev space $H^1(\Omega\ )$. For simplicity, we assume that we have zero Dirichlet boundary conditions. All elements of V and its subspace V^h vanish on $\partial\Omega$, the boundary of Ω . We introduce the triangulation of Ω in the following way. We first divide the region into non-overlapping substructures $\Omega_i\ , i = 1,\cdots,N$. To simplify the description of our algorithm, we confine our study to triangular (simplicial) substructures. In such a case, the original region must of course be a polygon (polyhedron). We also adopt the common assumption in finite element theory that all substructures are shape regular in the sense that H_i, the diameter of the Ω_i divided by the radius of the largest inscribed sphere in Ω_i is uniformly bounded. All the substructures Ω_i are divided into elements. The elements are also shape regular in the same sense. Since Schwarz-type domain decomposition algorithms use overlapping subregions, we extend each substructure to a larger region Ω_i'. We assume that the distance between the boundaries $\partial\Omega_i$ and $\partial\Omega_i'$ is bounded from below by a fixed fraction of H_i, and that $\partial\Omega_i'$ does not cut through any element. We make the same construction for the substructures that meet the boundary except that we cut off the part of Ω_i' that is outside of Ω.

We remark that some of the more intricate issues in the analysis of the Schwarz methods arise when the boundaries of the different subdomains Ω_i'

intersect at one or several points; cf. the discussion in Lions [13]. This happens for example if the region is L-shaped and is partitioned into two overlapping rectangles. We discuss this matter in our more complete paper on this subject.

Our finite element space is represented as the sum of N+1 subspaces

$$V^h = V_0^h + V_1^h + ... + V_N^h .$$

The first subspace V_0^h, which we also call V^H, is special. It is the space of continuous, piecewise linear functions on the coarse mesh defined by the substructures Ω_i. We use this space to provide a mechanism of global transportation of information similar to the role of coarse grid solvers in multigrid methods; cf. [8], [21] or [22]. It is quite easy to show that without such a device, any additive Schwarz method would converge no faster than the conjugate gradient method applied directly to the same coarse finite element discretization. The other subspaces are related to the subdomains, in the same way as the traditional Schwarz algorithm.

For self-adjoint elliptic problems, the finite element solution is the projection of the exact solution onto a finite-dimensional space. Different variants of Schwarz' algorithm, in particular the additive form considered in this paper, can also conveniently be described in terms of projections onto subspaces related to overlapping subregions which cover the region. These projections are orthogonal with respect to the symmetric bilinear form associated with the elliptic operator. The computation of the projection of an arbitrary function onto V^H involves the solution of a standard finite element linear system of algebraic equations which has on the order of N unknowns. This coarse, global approximation of the elliptic equation is of the same type as the local problems associated with the subdomains. It is often feasible to handle all these linear systems of equations by a direct method such as Gaussian elimination. The only real difference between the problem related to the first subspace and the others lies in the way the right-hand side of the linear system is generated as weighted averages with weights determined by the basis functions associted with the coarse mesh. We note that if we make the dimension of all the subspaces approximately equal, then we have $N+1$ linear systems, each with about than N unknowns, to solve in each step of the iterative solution of a linear system with about N^2 unknowns. In this algorithm two mesh sizes are used, but we could also introduce additional, intermediary finite element models.

The subspaces, V_i^h , which are related to interior substructures, are defined as $V^h \bigcap H_0^1(\Omega_i')$. The space H_0^1 is, as usual, the subspace of H^1

functions with zero trace. It is well known that this space can be extended continuously by zero to the rest of Ω. After such an extension, we can regard V_i^h as a subspace of V^h . We construct subspaces related to the boundary substructures similarly.

The projection $P_{V_i^h} = P_i$ is defined, for all of V^h, by the unique element of V_i^h, which satisfies

$$a(P_i v_h, \phi_h) = a(v_h, \phi_h), \; \forall \phi_h \in V_i^h. \tag{2}$$

Lions [13] has shown that the error propagation operator of the standard multiplicative variant of the Schwarz method can be written as

$$(I - P_2)(I - P_1),$$

for the case of two subregions. We can therefore view that algorithm as a simple iterative method for solving

$$(P_1 + P_2 - P_2 P_1) u_h = g_h,$$

with an appropriate right-hand side g_h . We note that the operator is a polynomial of degree two and thus is not ideal for parallel computing since two sequential steps are involved. If we use more than two subspaces this effect is further pronounced, even if the degree of the polynomial representing the multiplicative algorithm often is lower than the maximal. This is so because products of projections associated with subregions, which do not overlap, vanish.

The basic idea behind the additive form of the algorithm is to work with the simplest possible polynomial in the projections. We thus solve the equation

$$P u_h = (P_0 + P_1 + ... + P_N) u_h = g_h' \, , \tag{3}$$

by an iterative method. Since the operator P can be shown to be symmetric and positive definite, the method of choice is the conjugate gradient method. We must make sure that equation 3 has the same solution as equation 1, i.e. we must find the correct right-hand side. Since by equation 1, we have

$$a(u_h, \phi_h) = f(\phi_h) \, ,$$

we can construct the right-hand side g_h' by solving equation 2 for all values of i and by adding the resulting vectors. It is similarly possible to apply the operator P of equation 3 to any given element of V^h by applying each

projection P_i once and adding the results. Most of the work, in particular that which involves the individual projections, can be carried out in parallel.

It is well known that the number of steps required to decrease an appropriate norm of the error of a conjugate gradient iteration by a fixed factor is proportional to $\sqrt{\kappa}$, where κ is the condition number of P; see e.g. Golub and Van Loan [11]. We therefore need to establish that the operator P of equation 3 is not only invertible but that satisfactory upper and lower bounds on its eigenvalues can be obtained. We have proven the following theorem; cf. [8].

Theorem 1 *$\kappa(P)$ is bounded by a constant.*

Here, and elsewhere in this paper, the constants are independent of the diameters of of the substructures and the individual elements but they vary with the shape of the elements and substructures. Thus, in two dimensions, they depend on the minimum angle. As becomes apparent by examing the proof, the constant in the theorem grows if the overlap of the subregions shrinks.

We have discovered the additive algorithm independently, but its origin is not clear. We are quite interested in finding out about any previous work of this nature. We have found an algorithm quite similar to ours, in a paper by Nepomnyashchikh [18] on iterative substructuring methods on regions subdivided into a fixed number of subregions.

3 Iterative Refinement Algorithms

Finite element models on composite meshes can systematically be built up inside a conforming finite element framework. Conforming elements allow us to use a number of technical tools otherwise not available. We build our models systematically by first introducing a basic finite element approximation on the entire region and then repeatedly selecting subregions, and subregions of subregions, where the finite element model is further refined. We consider conjugate gradient algorithms where, in each iteration, problems representing relatively simple finite element models on the original regions, and the subregions, prior to further refinement are solved. We call these the standard problems. We thus use solvers for problems with uniform or relatively uniform mesh sizes, while the composite mesh can be strongly graded. As shown in [23], it is also possible to solve the standard problems inexactly. This type of algorithms is of interest when we wish to use adaptive meshes to increase accuracy locally. It is also important to note that existing industrial codes can be improved relatively easily by adding refinement regions in this manner; cf. [5], [9].

We study two families of algorithms which are multiplicative and additive respectively in the same sense as before. The additive algorithms are particularly well suited to parallel computing since, in each iteration step, we can simultaneously solve all the standard problems corresponding to all the different levels of refinement. Synchronization between the processors is required once in each conjugate gradient step when the residual is computed and assembled. One or a group of processors can therefore be assigned in a straightforward way to each of the standard subproblems.

We introduce some notations. Denote the original region Ω by Ω_1 and let $\Omega_i \subset \Omega_{i-1}, i = 1 \cdots k$, be subregions of subregions of the original region. As explained further in [23], we impose certain mild geometric conditions on $\Omega_{i-1} \setminus \Omega_i$. Let $V^{h_i}, i = 1 \cdots k$, be a family of conforming finite element spaces such that $V^{h_{i-1}} \bigcap H_0^1(\Omega_i) \subset V^{h_i} \subset H_0^1(\Omega_i)$. Our goal is to solve equation 1 using

$$V^h = V^{h_1} + ... + V^{h_k}.$$

The fundamental building blocks of our algorithms are the $P_j^i, i \leq j$, the projections onto the spaces $V^{h_i} \bigcap H_0^1(\Omega_j)$. We note that if $j > i$, then we solve a problem on Ω_j with a coarser mesh than if V^{h_j} were used. Such a problem is therefore relatively less expensive.

We can now describe a basic multiplicative refinement algorithm. The operator corresponding to this method is symmetric and we can accelerate

it with the standard conjugate gradient method. In each of $2k-1$ substeps of a basic iteration step, we compute the residual, using the just updated solution. The bilinear form corresponding to this residual now serves as a right-hand side. By solving in the subspace V^{h_i}, we compute the next correction as the projection of the current error onto this space. In each iteration, we carry out these steps in the order i= k, k-1, ..., 2, 1, 2, ... k-1, k. In [23], we establish the following result,

Theorem 2 *The multiplicative iterative refinement algorithm has a condition number which is independent of k and the number of degrees of freedom.*

When we now turn to additive algorithms, we note that the most natural algorithm would amount to using the projections P_i^i, defined above, in equation 2. However, it is relatively easy to show that the condition number of that algorithm grows linearly with k. A very promising method, for which to our knowledge the optimality has only been established in a quite special model case cf. [14], is based on using P_i^i - P_{i+1}^i , $i \leq k-1$ and P_k^k as the basic projections in equation 3. It can be shown that these differences of projections are projections and that the composite finite element space V^h is the direct sum of the corresponding subspaces. The difficulty experienced when trying to establish that the sum of these operators is uniformly well conditioned is related to the lack of flexibility in representing a given element of V^h as a sum of elements of the subspaces. If we however modify our projections, we arrive at an algorithm for which we have a proof of optimality.

Theorem 3 *The additive method defined by the projections* P_i^i - P_{i+2}^i, $i \leq k-2$, P_{k-1}^{k-1} *and* P_k^k *has a condition number which is independent of k and the number of degrees of freedom.*

We note that it is easy to show that the corresponding subspaces are larger than those of the previous method. This makes our proof possible. While the other method is believed to be faster, we have shown that the condition number of the method of Theorem 3 is bounded by twice the condition number of the other. For details see [23]. We also note that a convergence proof was outlined in [5] for the multiplicative algorithm in the case of k=2 and that proofs and discussions are given for both additive and multiplicative two level algorithms in [3] and [15].

References

[1] Ivo Babuška. The Schwarz algorithm in partial differential equations of mathematical physics. *Czech. Math. J.*, 83(8):328–343, 1958. In Russian.

[2] Ivo Babuška. Über Schwarzsche Algorithmen in partielle Differentialgleichungen der mathematischen Physik. *ZAMM*, 37(7/8):243–245, 1957.

[3] Petter E. Bjørstad. *Multiplicative and Additive Schwarz: Convergence in the 2 domain case.* Technical Report, Department of Computer Science, University of Bergen, Allégatan 55, Bergen 5000 , Norway, 1988. To appear in the proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations.

[4] Petter E. Bjørstad and Olof B. Widlund. Iterative methods for the solution of elliptic problems on regions partitioned into substructures. *SIAM J. Numer. Anal.*, 23(6):245–256, 1986.

[5] James H. Bramble, Richard E. Ewing, Joseph E. Pasciak, and Alfred H. Schatz. A preconditioning technique for the efficient solution of problems with local grid refinement. *Computer Methods in Applied Mechanics and Engineering*, 1988. To appear.

[6] Philippe G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, 1978.

[7] Maksymilian Dryja. A method of domain decomposition for 3-D finite element problems. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, 1988.

[8] Maksymilian Dryja and Olof B. Widlund. *An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions.* Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[9] Richard E. Ewing. *Domain Decomposition Techniques for Efficient Adaptive Local Grid Refinement.* Technical Report, Department of Mathematics, University of Wyoming, Laramie, Wyoming, 1988. To appear in the proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations.

[10] Kang Feng. Finite element method and natural boundary reduction. In *Proceeding of the International Congress of Mathematicans*, Warsaw, Poland, 1983.

[11] Gene H. Golub and Charles F. Van Loan. *Matrix Computations.* Johns Hopkins Univ. Press, 1983.

[12] Leslie Hart and Steve McCormick. *Asynchronous Multilevel Adaptive Methods for Solving Partial Differential Equations on Multiprocessors: Computational Analysis.* Technical Report, Comp. Math. Group, Univ. of Colorado at Denver, 1987. Submitted to Parallel Computing.

[13] Pierre-Louis Lions. On the Schwarz alternating method. I. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, 1988.

[14] Jan Mandel and Steve McCormick. *Iterative Solution of Elliptic Equations with Refinement: The Model Multi-Level Case.* Technical Report, Comp. Math. Group, Univ. of Colorado at Denver, 1988. To appear in the proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations.

[15] Jan Mandel and Steve McCormick. *Iterative Solution of Elliptic Equations with Refinement: The Two-Level Case.* Technical Report, Comp. Math. Group, Univ. of Colorado at Denver, 1988. To appear in the proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations.

[16] Steve McCormick. Fast adaptive composite grid (FAC) methods. In K. Böhmer and H. J. Stetter, editors, *Defect Correction Methods: Theory and Applications*, pages 115–121, Computing Supplementum 5, Springer-Verlag, Wien, 1984.

[17] Steve McCormick and Jim Thomas. The fast adaptive composite grid (FAC) method for elliptic equations. *Math. Comp.*, 46(174):439–456, 1986.

[18] S. V. Nepomnyashchikh. *On the application of the method of bordering for elliptic mixed boundary value problems and on the difference norms of* $W_2^{1/2}(S)$. Technical Report 106, Computing Center of the

Siberian Branch of the USSR Academy of Sciences, Novosibirsk, 1984. In Russian.

[19] H.A. Schwarz. *Gesammelete Mathematische Abhandlungen.* Volume 2, Springer, Berlin, 1890. First published in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, volume 15, 1870, pp.272–286.

[20] S.L. Sobolev. The Schwarz algorithm in the theory of elasticity. *Dokl. Acad. N. USSR*, IV(XIII):236– 238, 1936. In Russian.

[21] Olof B. Widlund. Iterative substructuring methods: Algorithms and theory for problems in the plane. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, 1988.

[22] Olof B. Widlund. Iterative substructuring methods:The general elliptic case. In *Computational Processes and Systems*, Nauka, Moscow, 1988. Proceedings of Modern Problems in Numerical Analysis, a conference held in Moscow, USSR, September , 1986. In Russian, also available from the author, in English, as a technical report .

[23] Olof B. Widlund. *Optimal Iterative Refinement Methods.* Technical Report, Department of Computer Science, Courant Institute, 1988. To appear in the proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations.

[24] Harry Yserentant. On the multi-level splitting of finite element spaces. *Numer. Math.*, 49:379– 412, 1986.

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